From demonstration to inquiry in teaching vectors with CAS

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IN TEACHING VECTORS WITH CAS

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Introduction

Noss and Hoyles (1996) 'open a window' on teachers using computers; they emphasize that the teachers are significant partners in the innovation process. Zehavi (1997) shows a specific example of an innovative approach to optimization problems, which was designed by teachers who learned to use CAS. This example is used to describe the changes that CASs bring to the mathematical environment of teachers. Tharp, Fitzimmons, and Brown-Ayres (1997) describe the practical relationships between teaching styles and teacher perception of accommodating graphical calculators in the class. In order to better understand the various ways for integrating CAS into teaching several researchers used a methodology of comparing pairs of teachers in their first CAS-in-teaching experience. Lumb, Monaghan, and Mulligan (2000) discuss the different experience in class of two teachers who started to use technology as a regular part of their mathematics courses, as well as the training project that was designed to support them. Kendal and Stacey (2001) provide a detailed comparative analysis of two pioneering teachers who adapted their teaching to use CAS with a focus on the teachers' beliefs about mathematics and mathematics understanding. This study was the basis for a plenary lecture by Stacey (2001) on *Teaching with CAS in a Time of Transition* in the 2nd CAME symposium, accompanied by a lecture by Zbiek (2001), *Influences on Mathematics Teachers' Transitional Journey in Teaching with CAS*. Zbiek reflected on the above studies that compared pairs of teachers and added finding from studies in her group. An important

factor is her work is training teachers by using task-based interviews as a technique to assess students' emerging understanding in a CAS environment.

In this paper we bring an example of the transition process in teaching Vectors with CAS. A high-school course on Vectors has the potential to integrate algebra and geometry in an advanced level. However, such integration has not been simple in the traditional mathematical environment. New teaching and learning opportunities emerge when Computer Algebra Systems (CAS) become part of the modern mathematical environment. The aim of the MathComp project at the Weizmann Institute is to broaden opportunities for learning and to promote greater mathematical understanding by using CAS. Within this framework, we have developed a portfolio for teachers – *Vectors in CAS*. The portfolio, distributed in electronic form on CD, contains 9 units - each of them includes *worksheets for students* (text files as well as *Derive* files) and an extended *teacher guide.* The teachers can use the materials as they find appropriate and they can also customize the portfolio to their preferences in teaching the topic.

Professional Development and Follow-up

The portfolio was introduced to 18 teachers who participated in a summer workshop and planned to teach the topic Vectors in the coming school year. We worked on selected activities from the 9 units and explained the rational, the goals, and the suggested teaching strategy for each activity. The activities can be classified into three main types: demonstration by the teacher using the teacher's computer, interactive files for explanation from the teacher's computer or by students in the lab, students' inquiry in the lab. The integration of the units in the teaching is flexible; besides the first unit, which introduces the syntax of the software for the basic terminology, algebraic and geometric interpretation of vectors, and operations with Vectors, the use of the other units is optional. During the school year we kept

in contact with the teachers by telephone discussions, email messages, and personal meetings. For each unit implemented by the teachers we asked them to prepare a reflective account.

In practice we found that four teachers did not use the portfolio at all due to logistics or lack of confidence in using the software. Two teachers admitted that they had mathematical difficulties in this topic and thus worked on the material only at home to gain a better understanding. Five teachers with a good command in the topic decided to use the material in a way that we did not plan, namely these teachers used the prepared files to design posters, transparencies or slides for demonstration without presenting the software to the class. As for two of them - when they felt more confident with the new mathematical environment they decided to use the interactive files for explanation in class. Seven teachers started by using the first unit for explanation in the class and then introduced the students to the software in the lab and worked on the interactive activities for explanation and practice; only four of them later dared to challenge the students to explore the guided inquiry activities. Encouraged by students' motivation these four teachers added new activities at various levels including inquiry, extension and reflection. In the verbal and written report of these teachers we detected comments of satisfaction such as "I feel as if I have been transformed into a different teacher". We describe below the transition process of one of these teachers MI.

The transition process of MI

We bring here the report of one teacher, MI, focusing on the teaching of the unit - *Geometrical Applications*. The mathematical activities of the units are described, accompanies by MI's comments (in italics).

MI told us: *Teaching vectors was a new experience for me a few years ago. I had a problem to convince myself, as well as my students, that it had any advantage over classical geometry. Gradually I realized that vectors as basic concepts, rather than lines and points, make it easier to grasp ideas such as division of a segment in some ratio, and hence the meeting of*
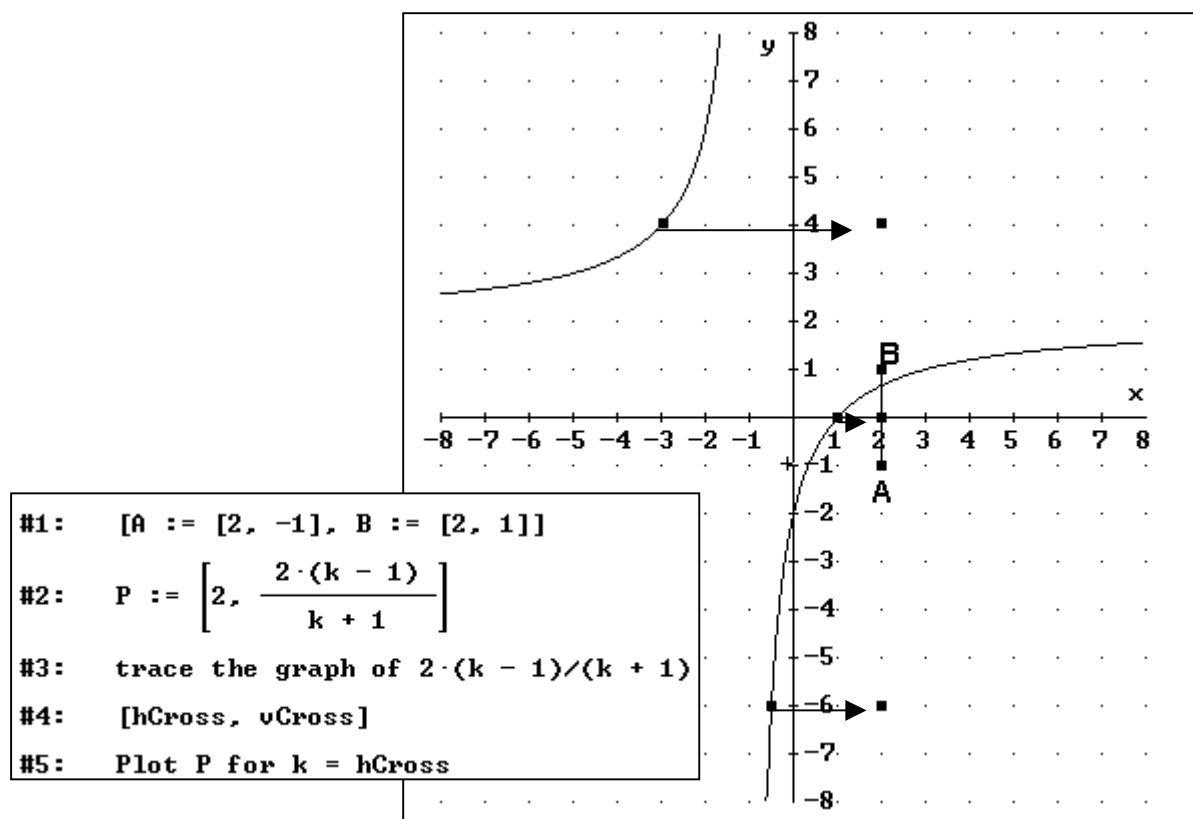
medians in any triangle and later on the scalar product that makes it easy to calculate the angle between two lines, and more. The main problems in trying to teach the subject was the cumbersome geometry involved in drawing vectors in an adequate way, and calculating quickly and accurately dot products and other manipulations. When we had the Vector-with-CAS workshop, I was thrilled to see how CAS makes us overcome the above problems. Then, I was afraid we have to face a new obstacle – the use of CAS by the students. However, in Class I found that our young students are better than their teachers are when it comes to adapting to the new technological environment.

The unit "Geometrical Applications" starts with interactive activities for finding a formula to divide a segment AB in ratio k:
$$P = \frac{A + B \cdot k}{k + 1}$$

The teachers rated highly the pedagogical potential of these activities, especially the task in which we see the graph of two given points A, B and several P_i points are marked on the line AB – one needs to find the k values for the P_i points and then add Q_i points near each of the P_i points.

MI: *The activity begins by solving a system of two linear equations to find the two coordinates of P and r leads the students to solve the equation P – A = k(B – P) for the unknown P. When we explored the formula, we learned that if k=n/d then P is the weighted mean of A and B – the denominator d is the weight of A and the numerator n is the weight of B. I found it difficult for the students to understand that the laws of addition and multiplication by a scalar are the same as in the case of real numbers, thus enabling us to get the formula in two steps! In order to help the students realize that in cases like this, speaking in the language of vectors makes a lot of sense, I decided to add a graphical interactive representation for exploring how the location of P is influenced by the value of k as shown in the following figure.*

```
#1:    [A := [2, -1], B := [2, 1]]

#2:    P := [2, 2·(k - 1)/(k + 1)]

#3:    trace the graph of 2·(k - 1)/(k + 1)

#4:    [hCross, vCross]

#5:    Plot P for k = hCross
```

The obtained formula for dividing a segment is then used for exploring geometrical theorems concerning combinations of two vectors in the plane: $\overrightarrow{CP} = a \cdot \overrightarrow{CA} + b \cdot \overrightarrow{CB}$. More specifically, we deal with the basic idea of a theorem and its inverse: The theorem states that in a triangle, any vector whose start point is one of the vertices and whose endpoint is on the opposite side of the triangle, is a linear combination of the two other sides, and the sum of coefficients is 1. The inverse theorem states that the endpoint of a linear combination of two sides of a triangle where the sum of coefficients is 1 must be on the opposite (third) side of the triangle.

The activities are designed for student inquiry, using the symbolic manipulator to check specific points and encouraging the students to obtain a formal proof, again using the Vector syntax of the software.

MI: *The instructions given in the worksheet were very clear, so I let the student work independently. I assumed that they will need help in formalizing the proof and was pleased that many of them came up with a clear proof of the first theorem. Here is what one student wrote:*

We start with the basic formula for division by the ratio of k:

$$P = \frac{A + B \cdot k}{k + 1}$$

$$\text{EXPAND}\left(P = \frac{A + B \cdot k}{k + 1}, \text{ Rational, A, B, P}\right)$$

$$P = A \cdot \frac{1}{k + 1} + B \cdot \frac{k}{k + 1}$$

P is a linear combination of A and B, and the sum of coefficients is 1:

$$\frac{1}{k + 1} + \frac{k}{k + 1} = 1$$

Thus we proved that if P is inside AB, then:

$$P = a \cdot A + b \cdot B$$

where:

$$a + b = 1$$

But:

$$C = a \cdot C + b \cdot C$$

Therefore:

$$P - C = (a \cdot A + b \cdot B) - (a \cdot C + b \cdot C)$$

$$P - C = a \cdot (A - C) + b \cdot (B - C)$$

However, the students needed help in organizing the proof of the inverse theorem. I presented the

following formal proof; but realized that the students could not actually see the difference between the

two proofs, thus could not appreciate the elegant logic expressed by the proof.

We start with the conclusion of the last proof

$$P - C = a \cdot (A - C) + b \cdot (B - C)$$
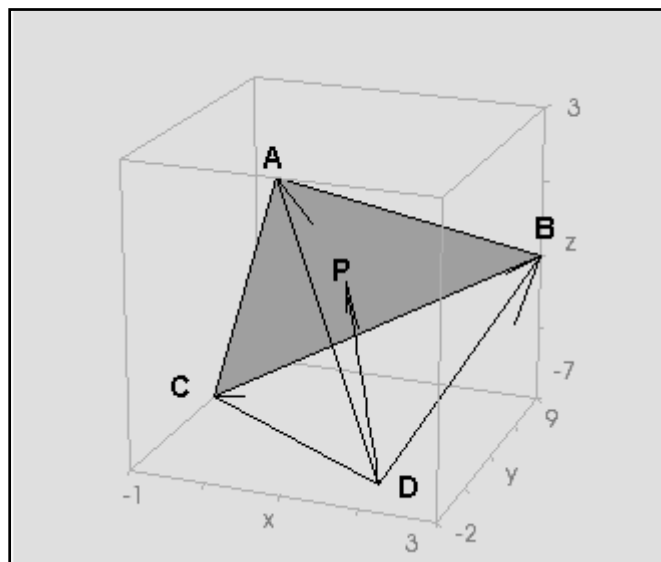
$$(P - C = a \cdot (A - C) + b \cdot (B - C)) + C$$

$$P = A \cdot a + B \cdot b - C \cdot (a + b - 1)$$

But a+b=1. Therefore:

$$P = a \cdot A + b \cdot B$$

Which proves that P is inside AB!

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Next, the analogical case in the 3D-space $\overrightarrow{DP} = a \cdot \overrightarrow{DA} + b \cdot \overrightarrow{DB} + c \cdot \overrightarrow{DC}$ $(a + b + c = 1)$, is dealt with, leading to finding the center-of-gravity of a triangle embedded in 3D-space. In previous activities we learned that a point inside a segment is the weighted mean of its endpoints. The greater the weight of an endpoint the closer is the point P to that endpoint. How does one generalize the observation to a point inside a triangle? A point inside a triangle is the weighted mean of its vertices, the sum of weights is 1. In particular the midpoint (center of gravity) of a triangle has equal weights (1/3 each), as the midpoint of a segment, which has equal weights (1/2 each).

MI: *Here again I enjoyed the freedom of approaching the subject from a new viewpoint. I found that I get there via two ways. One way, suggested in the given worksheet, will be an algebraic way, which in essence, gives each point its baricentric coordinates. The greater the weight of a vertex, the closer is the point to this vertex. Based on students' difficulties in the 2D case I thought it would be helpful to add a second way that has a more geometrical flavor.*

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```
[A := [-3, -1], B := [3, -2], C := [1, 2]]
O := [0, 0]
```

Could we write O as a linear combination of A, B, C?

An algebraic solution would be to solve the equation:

```
O = a·A + b·B + c·C
SOLVE(O = a·A + b·B + c·C, [a, b, c], Real)
         9·a - 8·c = 0 ∧ 9·b - 5·c = 0
```

We did not get an explicit solution, becase we had only 2 equations!

Let's add a third equation - the sum of coefficients is 1

```
O = a·A + b·B + c·C ∧ a + b + c = 1
SOLVE(O = a·A + b·B + c·C ∧ a + b + c = 1, [a, b,
  c], Real)
```

$$a = \frac{4}{11} \wedge b = \frac{5}{22} \wedge c = \frac{9}{22}$$

The second way is to use twice what we know about a point inside a segment.

Let's try a geometric approach, based on the previous paragraph.

First, we find the intersection point, P, of AB and CO

```
A + t·(B - A) = C + s·(O - C)
SOLVE(A + t·(B - A) = C + s·(O - C), [s, t], Real)
```

$$s = \frac{22}{13} \wedge t = \frac{5}{13}$$

```
P = A + t·(B - A)
P = (1 - t)·A + t·B
```

$$P = \frac{8}{13}·A + \frac{5}{13}·B$$

Since O is inside CP, there are c and p such that:

```
O = c·C + p·P ∧ c + p = 1
SOLVE(O = c·C + p·P ∧ c + p = 1, [c, p], Real)
```

$$c = \frac{9}{22} \wedge p = \frac{13}{22}$$

If we substitute in the previous equations we get:

$$O = \frac{13}{22}·\left(\frac{8}{13}·A + \frac{5}{13}·B\right) + \frac{9}{22}·C$$

$$O = \frac{4·A}{11} + \frac{5·B}{22} + \frac{9·C}{22}$$

We see that the weights of the vertices are 8/22, 5/22, and 9/22 for A, B and C. Now we may easily explore how the coordinates change when one goes to the sides of the triangle (the coordinate of the vertex opposite it is zero and the sum of the two other coordinates is 1), and then outside it (here one finds six different zones).

In general, if P is the midpoint of AB, then:

$$P = \frac{1}{2} \cdot A + \frac{1}{2} \cdot B$$

Any point Q on CP obays the equation:

$$Q = (1 - c) \cdot P + c \cdot C$$

$$Q = (1 - c) \cdot \left[\frac{1}{2} \cdot A + \frac{1}{2} \cdot B \right] + c \cdot C$$

$$Q = \frac{1 - c}{2} \cdot A + \frac{1 - c}{2} \cdot B + c \cdot C$$

So that A and B have the same weights.
Let Q be the meeting point of two medians, then all weights must be equal:

$$c = \frac{1 - c}{2}$$

$$\text{SOLVE} \left(c = \frac{1 - c}{2}, \ c, \ \text{Real} \right)$$

$$c = \frac{1}{3}$$

The meeting point of all three medians is

$$\frac{1}{3} \cdot A + \frac{1}{3} \cdot B + \frac{1}{3} \cdot C$$

From the account of MI we can follow how he appreciated the transition from a doubtful teacher regarding both teaching Vectors and using technology, to a creative and imaginative user of technology for this topic. We appreciated, and learned from, his ability to use the new environment to integrate and generalize mathematical and pedagogical ideas.

Summary

Teaching Vectors with CAS has clear advantages; the computer can handle the algebraic manipulations as well as providing visual interpretation in 2-D and 3-D space. Moreover, proofs using Vector operations are short and elegant but they are not intuitive. The graphical-

symbolic environment can be used for guided experiments and inquiry to facilitate understanding. Students have the opportunity to express and develop their ideas, while teachers are motivated to design further activities to suit the needs of the students and their own beliefs about mathematical understanding and teaching mathematics with CAS (as was found by Zehavi, 1997, Lumb et. al, 2000, and Kendal & Stacy, 2001). Based on the feedback from the teachers we have modified the portfolio. It includes activities that the teachers, our *significant partners*, created.

References

Kendal, M. & Stacey, K. (2001). The impact of teacher privileging on learning differentiation with technology. *The International Journal of Computers for Mathematical Learning*, 6(2), 143-165.

Lumb, S., Monaghan, J., Mulligan, S. (2000). Issues arising when teachers make extensive use of a Computer Algebra'. *Int. J. of Computer Algebra in Mathematics Education.* 7(4), 223-240.

Noss, R. & Hoyles C. (1996). *Windows on Mathematical Meanings: Learning Cultures and Computers*. Dordrecht: Kluwer.

Stacey, K. (2001). Teaching with CAS in a time of transition. *Web Proceedings of the 2nd CAME Symposium.* Utrecht: The Netherlands.

http://ltsn.mathstore.ac.uk/came/events/freudenthal/

Tharp, M.L., FitzSimmons, J.A., & Brown-Ayers, R.L. (1997). Negotiating a technological shift: Teacher perception of the implementation of graphing calculators. *Journal of Computers in Mathematics and Science Teaching, 16*(4)*, 551-575.

Zbiek, R. M. (2001). Influences on mathematics teachers' transitional journeys in teaching with CAS. *Web Proceedings of the 2nd CAME Symposium.* Utrecht: The Netherlands. http://ltsn.mathstore.ac.uk/came/events/freudenthal/

Zehavi, N. (1997) Changes that computer algebra systems bring to teacher professional development. In (E. Pehkonen , ed.) *Proceedings of the 21st Conference of the IGPME* (pp. 307-314). Lahti: University of Helsinki.