Cases for CAS in Teaching, Learning and Assessment

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Abstract

An international T³-project group is working on a book. The goal is to present convincing cases for the use of CAS on the secondary II level.
Two examples from different school levels of such CAS-Cases will be presented.
1. I will show an elementary proof of Heron's formula made easy with CAS. Furthermore I will present new examples using Herons formula as a functional module dynamically, leading to locus considerations comparable with Cabri.
2. Examples of a final assessment: The first example shows how classical curve discussions, which are trivialised with CAS, can be extended to challenging exercises, which are not solvable without CAS. The second example is about the cross and dot product using polar coordinates: Does a plane, flying on the "axis of evil" between Pyongyang and Karthoum, cross Iraq, where is the northernmost point of the flight path etc.

1. Example: Heron's Formula

The following extract of a tested teaching sequence will show you
*   *how students are able to prove Heron's formula, a rather hard task for most of the students without a CAS-calculator,*
*   *how Heron's formula can be used as a functional module (functional thinking),*
*   *how the usual pure calculation exercises about Heron's formula can be replaced by nice and pretentious ones, which are leading to reflection and deepening of mathematics.*

Worksheet: Heron's Formula
A triangle can be constructed with three sides given; therefore its area is also determined. Not everything which is constructable, has to be calculated. But in this case the Greek mathematician Heron of Alexandria found the following formula for calculating the area of a triangle from its sides:
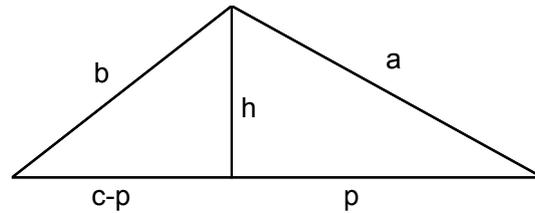

Heron's formula for the area of a triangle

$$area = \sqrt{s(s-a)(s-b)(s-c)} \text{ with } s = \frac{a+b+c}{2} \text{ (s = half of the circumference)}$$

Exercise 1: Store the above formula as a function named *heron(a,b,c)*. (attach the additional condition with |)
Calculate the area of the rectangular triangle with sides 3, 4, 5.
Calculate the area of some other triangles (choose the length of its sides yourself). Interpret possible error messages.
Calculate the side *c* of a triangle from *area* = 5, *a* = 3, *b* = 4.

Exercise 2:
Prove Heron's formula using the figure and the following guidelines (insert your results in the worksheet)!

Proof of Heron's formula

Follow the instructions:

a) We calculate the square of Heron's formula *area^2 = heron(a,b,c)^2* in order to avoid roots. The CAS-calculator returns (*s* has been replaced by *a*, *b* and *c*):

$$area^2 = \frac{-(a+b+c)(a+b-c)(a-b-c)(a-b+c)}{16}$$

which has to be proven.

b) For the area of the triangle, we start with a well-known formula using baseline *c* and height *h*:

$$area^2 = \frac{c^2 \cdot h^2}{4}$$

c) *c* is given. Express *h^2* first by *a* and *p* and then by *b* and *c-p* using the Pythagorean theorem:

$h^2 =$ (using *a* and *p*, store it in *h2*)

$h^2 =$ (using *b* and *c-p*)

d) Equate both expressions for *h^2* and solve it with respect to *p*.

Check your answer: $p = \dfrac{a^2 - b^2 + c^2}{2c}$

e) Factor *h2* of c) and insert *p* from d) (CAS: *factor()|p=..*). Store the result again in *h2*

(= *h^2*) and insert it in $area^2 = \dfrac{c^2 \cdot h2}{4}$ from b).

Compare with the expression in a).

Exercise 3: Calculate the area of a trapezoid with sides a = 6, b = 4, c = 3, d =3. Define generally a functional module trapez(a,b,c,d).

Exercise 4: Does a triangle exist with sides 3, *x*, 2*x* and area *T*=10?
For which given area *T* does exist a solution? Try also with *T* = 1, 3, 0.
Draw the function *y1(x)=heron(3,x,2x)* and determine roughly its range of definition and range of values.
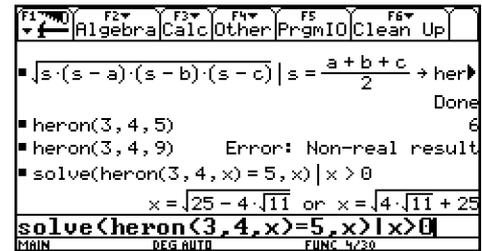Draw the vertices *C* of possible triangles *ABC* with |*AB*|=3. On which curve are these vertices lying?
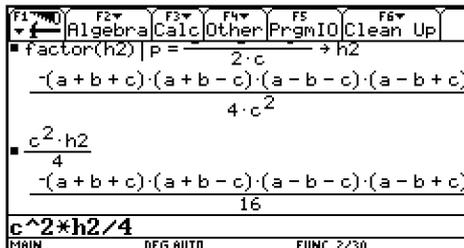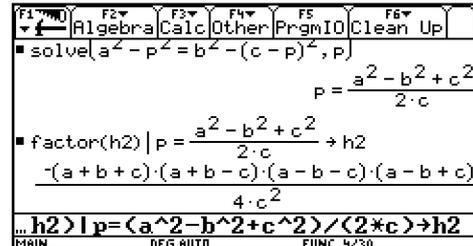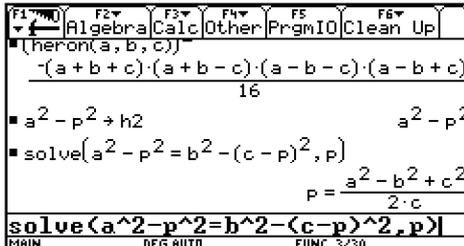You can also construct the locus using Cabri.

Solutions to the worksheet

Exercise 1

If one side is bigger than the sum of the others, a real solution doesn't exist. The latter example shows you, that there exist at most 4 solutions to the equation, whereas two of them are corresponding to a triangle ($x > 0$).

Exercise 2: You can see the steps of the proof on the screenshots below.
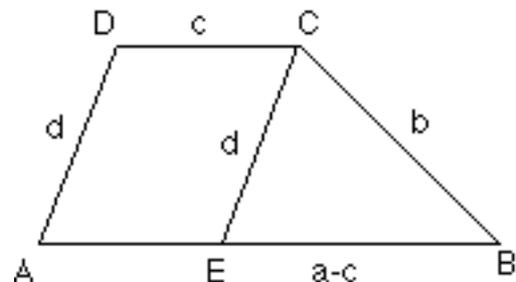
The proof with CAS is much easier than by hand, because the most difficult part is to transform the last expression to the shape of Heron's formula. CAS simplifies Heron's formula automatically to the last expression.

Exercise 3

We calculate the height of the trapezoid using the triangle EBC.
solve(heron(abs(a-c),b,d)
=h*abs(a-c)/2,h)

Then we calculate the area by $\dfrac{(a+c)h}{2}$.

This general formula is very extensive and can be stored as a functional module
trapez(a,b,c,d), e.g. trapez(3,3,6,4)=6√5
The function fails for rectangles!

Exercise 4

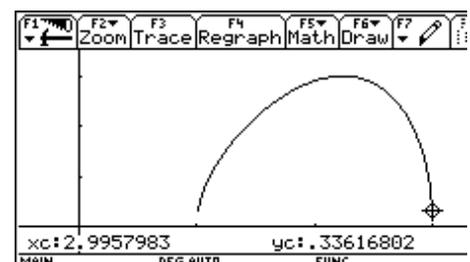*solve(heron(3,x,2x)=10,x)*
result: *false*
For *T* = 1, 3, 0 you get 1.109 or 2.962, ±√5, 1 or 3 (positive values only).
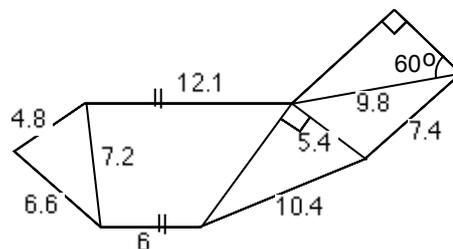With trace or the function table you get the range of definition [1,3] and the range of values [0,3].

The vertices *C* of possible triangles with *A* = (0;0) and *B* = (3;0) are lying on the Apollonian circle over the segment $\overline{AB}$ with ratio 1:2. This is the circle of Thales over the segment with endpoints (2;0) and (6;0).

Note: This exercise can be done at an elementary level (9th school year). I did it after having introduced the Apollonian circle as an application of the intercept theorems. It can also be used as an experimental introduction to the Apollonian circle. In the book you will find more exercises of this kind.

Exam questions (in the book you will see more of them)

1. Find the area *T* of the triangle with vertices *A* = (13;2), *B* = (5;17) and *C* = (22;-4), using Heron's formula.

2. In a consolidation of property a farmer can exchange his agricultural land (see figure beside, measures in km) for a piece of land of quadratic shape. What is the extension of the quadratic piece of land?
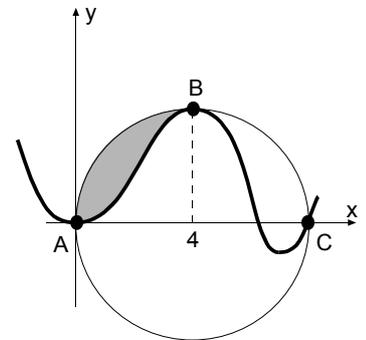

Solutions:

1. Store the vertices of the triangle: *[-13,2]->a, [5,17]->b, [22,-4]->c. heron(norm(b-a),norm(c-b),norm(c-a))*, result: 316.5.

2. We suppose that the modules *heron(a,b,c)* and *trapez(a,b,c,d)* are still stored, otherwise students have to define them again.
   First we calculate the lacking side *s* of the trapezoid: *10.4^2-5.4^2->s*.
   The rest is an application of the functions *heron* and *trapez*:
   √(heron(4.8,7.2,6.6)+trapez(12.1,7.2,6,s)+5.4*s/2+heron(5.4,7.4,9.8)+4.9^2*√(3)/2)
   result: 12.028 km.

2. Example of a final Exam in Switzerland

In Switzerland every canton has its own educational system. Therefore no centralised final examinations exist i.e. each teacher usually arranges his own examination. However all of them are published to assure quality. The following final exam questions are merely examples of many variants. In the book you will find the entire exam.

Question 1

a) Determine a polynomial *f* with smallest possible degree, which satisfies the following conditions: The graph (does not have to correspond to the sketch)

   - has horizontal tangent in the origin,
   - touches the circle *k* with radius 4 and centre (4;0) in *B* and
   - has an inflection point in *C*.
   - The shaded area amounts to 6 (integral between *A* and *B*).

b) Outline the real curve (incl. circle).
   Tip: Accelerate the calculation by using a numeric value in the function.

c) Choose a parameter *k* (instead of 6) for the integral between *A* and *B*. Can *k* be arbitrary? (Give a reason for your answer.) If the answer is yes, outline the curve for *k* = 0 (incl. circle).

d) Discuss the asymptotic behaviour for $x \to \pm\infty$ in dependence of *k*.

4. An plane flies on a greater circle in 10 km height around the earth on the "axis of evil". It passes Pyongyang in North Korea (126^0 east; 39^0 north) and Kharthoum in Sudan (33^0 east; 16^0 north). Earth radius r = 6371 km.

a) The aeroplane needs 20 hours from Pyongyang to Kharthoum. Which is its average speed?

b) At which point (geographical length) does the plane pass the equator for the first time after Khartoum?

c) Which is the northernmost point of the flight path (Cartesian coordinates are sufficient)?

d) Does Iraq lie on the "axis of evil"?
   Iraq extends from latitude 27^0 to 37.5^0 north on the meridian 44^0 east through Baghdad.

Solution to question 1

Solution to a) and b):

Establish the conditions by hand:

$f(0) = 0$, $f\,'(0) = 0$, $f(4) = 4$, $f\,'(4) = 0$, $f(8) = 0$, $f\,''(8) = 0$, $A = 4\pi - \int\limits_0^4 f(x)\ dx = 6$

Determine the polynomial due to the number of conditions (here 7):

$\qquad f(x) = a^*x^6 + b^*x^5 + c^*x^4 + d^*x^3 + e^*x^2$

To save calculator time the last to constants were set to zero (according to the first two conditions).

Now we determine the first and second derivatives and store them in $f1(x)$ and $f2(x)$. We store the remaining five equations above in $e1$, $e2$, $e3$, $e4$ and $e5$ and solve the corresponding system of equations:

$\qquad$ solve(e1 and e2 and e3 and e4 and e5, {a,b,c,d,e})

The result appears in the form $a =...$ and $b =...$ and .... $e =.....$
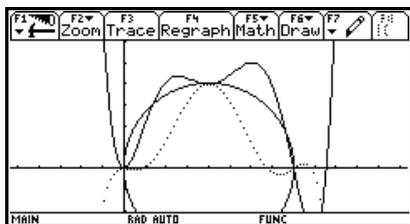
We store $f(x)|result$ in $y1(x)$ and we get

$\qquad y1(x) = -0.001x^6 + 0.036x^5 - 0.300x^4 + 0.930x^3 - 0.587x^2$

We draw the function (using numeric values to accelerate the drawing process)

We also can draw the circle (as a two-piece function or roughly with *F7 Circle* in the graph window, or with *Circle 4,0,4* in the Home Screen)

Solution to c):

We modify the last equation and replace 6 by k and solve the system of equations again (grabbed from history). One sees that in the new function k appears in linear terms only and therefore the system is always solvable. We store the function in $y2(x)$ and draw it for $k = 0$ (see figure, dashed for $k = 6$).

Solution to d):

The coefficient of x^6 is $a = -0.000657(k\text{-}3.766)$, thus

$k > 3.766 \quad\Rightarrow\quad \lim\limits_{x\to\pm\infty} f(x) = -\infty$

$k < 3.766 \quad\Rightarrow\quad \lim\limits_{x\to\pm\infty} f(x) = +\infty$

$k = 0 \quad\Rightarrow\quad \text{Coefficient of } x^5 < 0 \quad\Rightarrow\quad \lim\limits_{x\to\infty} f(x) = \infty \text{ and } \lim\limits_{x\to-\infty} f(x) = -\infty$

Note: Introduction of parameters leads to a new dynamic in algebra. With computer programs you can vary the parameter k with sliders and change the graph of the function interactively. In geometry this interactivity is well known through dynamic geometry programs. Of course, the size of the area is not the only number, which can be parameterised. In each equation numbers can be replaced by parameters. By doing so you get unlimited possibilities for individual and group activities and their presentation in form of animations.

Student solution to question 1

3)

a) Bedingungen:
1. $f(0) = 0$ ✓ ⇒ $g = 0$
2. $f'(0) = 0$ ✓ ⇒ $f = 0$
3. $f(4) = 4$ ✓
4. $f'(4) = 0$ ✓
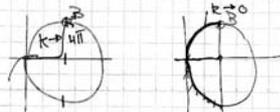5. $f''(8) = 0$ ✓
6. $f''(8) = 0$ ✓
7. $\int_{0}^{4\pi} f(x)\,dx = 6$ ✓

⇒ Polynom 6. Grades: $f(x) = ax^6 + bx^5 + cx^4 + dx^3 + ex^2 + fx + g$

⇒ TR: $f(x) = -0.001468x^6 + 0.035892x^5 - 0.3x^4 + 0.93x^3 + 0.5813x^2$

b)

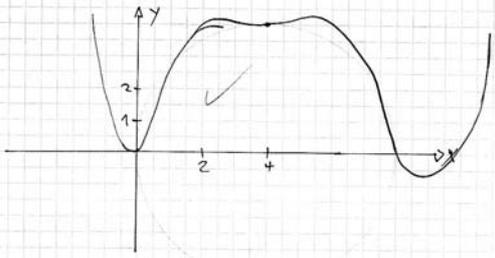

c)
$a = -0.00065\bar{7}(k - 3.7664)$
$b = 0.01665(k - 3.8445\bar{7})$
$c = -0.1507(k - 4.00823)$
$d = 0.5749(k - 4.3824)$
$e = -0.785256(k - 5.25208)$

k kann zwischen 0 und 4π liegen.

Negativ darf k nicht sein, sonst geht die Kurve nicht durch B!
$k = 0$:

d) Parameter a (bei höchster Potenz x^6) $a = -0.00065\bar{7}(k - 3.7664)$
Wenn $k < 3.7664$ ⇒ $x \to \infty : y \to \infty$
               $x \to -\infty : y \to \infty$ ✓
Wenn $k > 3.7664$ ⇒ $x \to \infty : y \to -\infty$
               $x \to -\infty : y = -\infty$ ✓
Wenn $k = 3.7664$ $a = 0$ ⇒ $x \to \infty : y = -\infty$
  x^6 fällt weg. $x \to -\infty : y = \infty$ ✓

9½

Solution to question 2

a) Store the polar coordinates as a (vector) function of three variables, namely radius *r*, degree of longitude *x* and degree of latitude *y*:
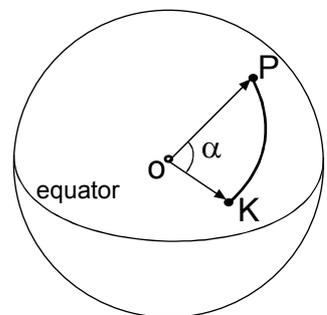
$$\vec{v}(r,x,y) = \begin{pmatrix} r\ \cos(x)\ \cos(y) \\ r\ \sin(x)\ \sin(y) \\ r\ \sin(y) \end{pmatrix}$$

with CAS: *[rcos(x)cos(y),rsin(x)cos(y),rsin(y)]->v(r,x,y))*

Define the position vectors for Pyongyang *P* and Karthoum *K*. To avoid big numbers we take *r* = 1 (unit sphere).

$$\overrightarrow{OP} \parallel \vec{p} = v(1, 126^0, 39^0) \qquad \overrightarrow{OK} \parallel \vec{k} = v(1, 33^0, 16^0)$$

Store these vectors with numerical values. Switch now to radian mode and calculate the angle between $\vec{p}$ and $\vec{k}$:

$$\alpha = \arccos\frac{\vec{p} \cdot \vec{k}}{|\vec{p}|\,|\vec{k}|} \underset{CAS}{=} 1.436. \text{ Hence } v = \frac{s}{t} = \frac{6381 \cdot \alpha}{20} = 458.163\ \frac{km}{h}.$$

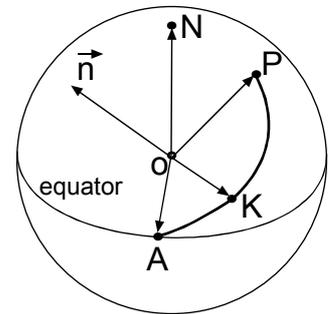b) Let $\vec{n} = \overrightarrow{OK} \times \overrightarrow{OP}$ be a normal vector to the greater circle through Pyongyang and Karthoum and *A* the intersection point of this circle with the equatorial circle. Then we have the following conditions:

$$\overrightarrow{OA} \perp \vec{n} \text{ and } \overrightarrow{OA} \perp \vec{e_3} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \Rightarrow \overrightarrow{OA} \parallel \vec{e_3} \times \vec{n}$$

If we take the unit vector $\vec{e_A} = \dfrac{\overrightarrow{OA}}{\left|\overrightarrow{OA}\right|}$, its first component will just be the cosine of the searched angle, therefore we get the longitude of point A (switch to degree mode):

$$\varphi = \arccos(1^{st} \text{ component of } \vec{e_A}) \underset{CAS}{=} \arccos(\vec{e_A}[1,1]) = 13.854^0$$

c) Let *H* be the highest (northernmost) point of the flight path. We have the following conditions:

$$\overrightarrow{OH} \perp \vec{n} \text{ and } \overrightarrow{OH} \perp \overrightarrow{OA} = k\vec{e_A} \Rightarrow \vec{h} = \vec{n} \times \vec{e_A} \text{ is a vector in direction of } \overrightarrow{OH}.$$

Hence $\overrightarrow{OH} = 6381\dfrac{\vec{h}}{\left|\vec{h}\right|} = \begin{pmatrix} -1150.281 \\ 4664.148 \\ 4199.968 \end{pmatrix}$

d) Store the unit vector $\vec{i} = v(1, 44^0, x)$ of the Iraqian meridian.

The condition for the direction to the "axis of evil" is $\vec{n} \times \vec{i} = 0$.
This equation solved with respect to *x* gives: $x = 180^0(k+0.1317)$.
For *k* = 0 we get $x = 23.705^0$, which means that the "axis of evil" passes south of Iraq.

Note: *Using defined functions as a module makes CAS-calculators very efficient and allow tackling realistic problems. Both tasks above would be very difficult to solve by hand or with scientific calculators. Tasks of this kind look very difficult at a first glance. But with some training they become standard problems.* It is certainly of interest to know which exercises and examinations preceded this final exam. In the book you will find the whole "history" to question 2.