From symmetry as a property to reflection as a geometrical transformation :evolving meanings and computational tools

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Programa de Estudos Pós-Graduados em Educação Matemática
PUC, São Paulo

Abstract: *This article reports on a study aiming to design learning systems in which students' knowledge of reflection is brought closer to socially-accepted, institutional knowledge of this isometry and to compare how their activities shape and are shaped by different forms of mediation. It presents descriptions of the interactions of groups of students (aged 12-13 years) with one of two computational microworlds, based on either dynamic geometry or multiple-turtle geometry, during attempts to construct and validate a tool for reflection and considers how the tools of the microworlds were important in mediating the passage between meanings emphasising reflection as property and those emphasising reflection as function.*

INTRODUCTION

Although research relating to the learning of transformations is still relatively under-represented in the mathematics education literature, a considerable number of studies have addressed students' views of reflection in plane geometry in a variety of different learning systems, including paper-and-pencil tasks (e.g. Küchemann, 1981; Grenier, 1988; Bell, 1993) as well as physical devices and activities (see, Lehrer *et al.,* 1998; Zuccheri, 1998) and computer-based tools (for example, Thompson, 1985; Gallou-Dumiel, 1987; Edwards 1992; Leron & Zazkis, 1992; Malara, 1995 Hoyles & Healy, 1997). These studies confirm that the majority of learners know quite a lot about the properties of the reflection transformation. Nearly all know that objects and their images are congruent and most also construct some distance between the axis of reflection and both image and pre-image as equal. This knowledge does not, however, correspond to its precisely definable mathematical counterparts and is operationalised differently depending on particular features of the task and the tools available during task resolution.

Generally speaking, students' have more success on tasks that can be interpreted from what Piaget and Garcia (1989) describe as an *intrafigural* perspective, where attention is directed towards the internal relationships of figures, than on tasks involving *interfigural* demands requiring attention to the relationships by which figures are associated with aspects external to them. In terms of the three views of

geometrical transformations distinguished by Grenier and Laborde (1988), it seems that most students associate the reflection transformation with a relationship between two geometrical configurations or two parts of the same configuration rather than with a mapping of figure-objects onto figure-images or of a plane onto itself. The predominate – and essentially intrafigural – view, hence places little emphasis on the functional character of the transformation.

These considerations motivated a study with the dual concern of (i) building learning systems which would enable students to build from views of reflection based on internal (intra) properties of figures to views taking account of external (inter) relationships between figures and would emphasise a functional perspective on the transformation while (ii) investigating how the meanings students forge in activity are shaped by – and shape – different mediation.

<center>THE STUDY</center>

To address this dual concern, the study comprised of two phases, the *design phase* and the *comparison phase*. During the *design phase*, learning systems were developed using an iterative methodology that cycled between the elaboration of tools and tasks and observation of students interacting with them. Computer microworlds were chosen as arenas for activity in the learning system because interacting with them involves users in extending a given tool-set (or primitives) to create their own computer-based models. It was hypothesised that this activity would involve students with functional aspects of reflection.

Different microworlds offer access to different modes of interacting with different mathematical models of geometry and, to compare how different means of expressing mathematics shape and become shaped by the developing understandings of those who use them, the learning systems incorporated one of two computational microworlds. One was based on the dynamic Euclidean geometry of *Cabri-Géomètre* (DEG) with which interactions involve direct manipulation of a model of the theoretical field of Euclidean geometry. The other was based on multiple turtle geometry as represented in the *Microworlds Project Builder* version of the programming language Logo (MTG). MTG interactions involve the programming of multiple turtles, whose movements around a two-dimensional surface are controlled by symbolic code.

A set of five tasks was developed for use in the learning systems with the condition that, as far as possible, the mathematical demands remained consistent regardless of which microworld was in use. These demands began with the construction of
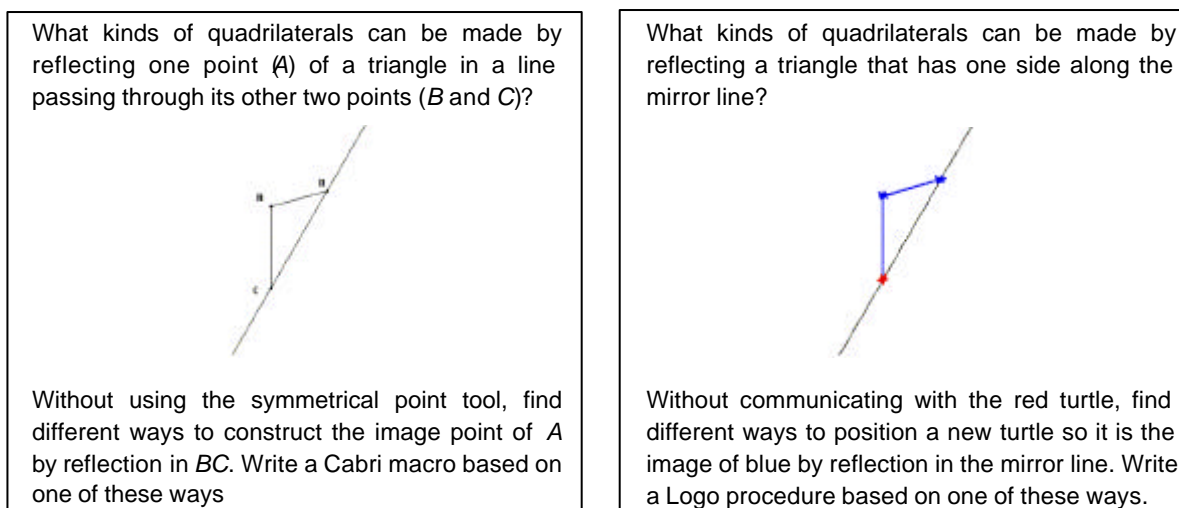
symmetrical figures and culminated in an activity designed to encourage students to focus on the plane (and the objects contained in it) as a point/turtle-set(s) and enact reflection as a mapping of this set onto itself.

In the *comparison phase*, data was collected during five ninety-minute sessions, as twenty-four students (aged between 12 and 13 years) interacted with these microworlds and tasks. The students were divided into four groups of six. Two groups worked with the DEG microworld and two with the MTG microworld. The researcher also acted as a participant in the groups, assuming the role of the teacher. The students were selected on the basis of their responses to a paper-and-pencil questionnaire, so that response profiles of students were similar across groups and each group consisted of students across the achievement range representative of the inner-city school in which they studied. During computer interaction, students worked in pairs. Audio recording of all the group interactions were made and transcripts of these, along with students' computer constructions and written works, as well as researcher notes made during and immediately following each session, comprised the data for analysis.

To illustrate the results of both phases, the remainder of the paper will concentrate on one of the five tasks included in the learning systems, an activity that required students to build their own computational tool that could be used to produce the images under reflection given any pre-image point (or turtle) and any axis.

FROM SYMMETRICAL FIGURES TO A TOOL FOR REFLECTION

Figure 1 presents the DEG and MTG versions of the third task students encountered during microworld interactions. This task was modified from one applied in a Logo context by Graf (1988). In its previous form, the task had involved discovering the types of quadrilaterals that could be produced when a triangle is reflected in one of its three sides (an intrafigural activity), using given microworld tools to construct the image. For this study, the task had a second part in which students were to use the properties of the quadrilateral to define their own tools for constructing images of points or turtles under reflection. In this way, it was hoped that any intrafigural relationships would be revisioned from an interfigural perspective.
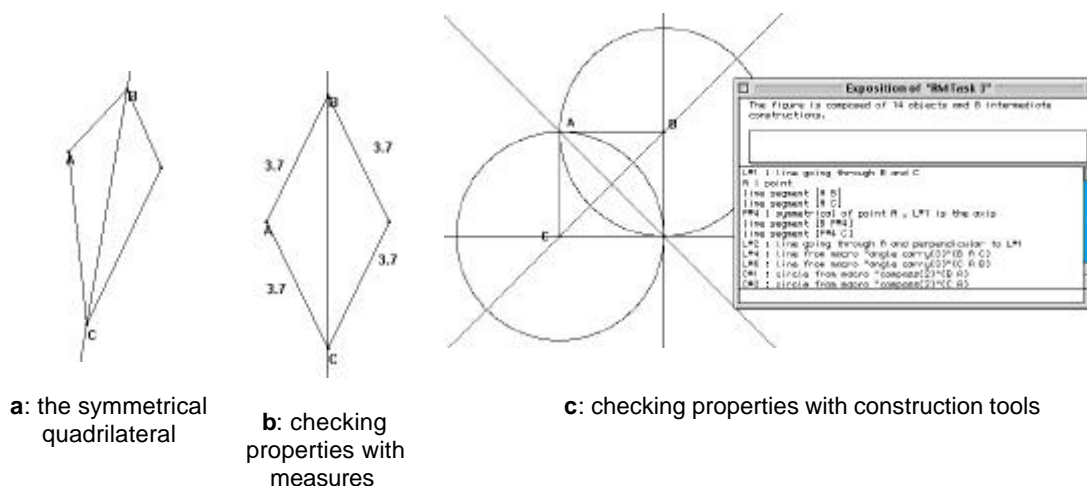
What kinds of quadrilaterals can be made by reflecting one point (*A*) of a triangle in a line passing through its other two points (*B* and *C*)?	What kinds of quadrilaterals can be made by reflecting a triangle that has one side along the mirror line?
Without using the symmetrical point tool, find different ways to construct the image point of *A* by reflection in *BC*. Write a Cabri macro based on one of these ways	Without communicating with the red turtle, find different ways to position a new turtle so it is the image of blue by reflection in the mirror line. Write a Logo procedure based on one of these ways.

Figure 1: The DEG and MTG versions of Task 3

Interactions with the DEG microworld

The DEG microworld was based on the first Macintosh version of Cabri-géomètre. In addition to the default primitives, a number of additional tools were available under the construction menu. This included two tools for constructing equal lengths, `compass(2)`[1] and `compass(3)` and a set of tools for constructing equal angles `angle carry(3)`, `angle carry(4)` and `angle carry(5)` – the number in brackets signifying the number of points to be specified as inputs. During the first two tasks, students had been introduced to these tools, as well as tools for constructing parallel and perpendicular lines and, where requested, measuring tools.

In both the groups participating in the comparison phase of the study, all the pairs began the task by applying the `symmetrical point` tool to point *A* of the triangle, using the line passing through points *B* and *C* as an axis of reflection (Figure 2a). Particular types of quadrilaterals were attempted by dragging the vertices *A*, *B* and *C* as students engaged in the process of trying to soft-construct squares, rectangles, rhombi and parallelograms. Properties of the quadrilaterals were made explicit either by adding measures (Figure 2b) or using the construction tools (Figure 2c). As different quadrilateral types were attempted, students could see which of these properties held for all configurations of the symmetrical quadrilateral. Not all the students isolated as comprehensive a set of properties as the pair who produced Figure 2c, but all identified, at least, the two pairs of adjacent sides of equal length.

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[1]   The `compass(2)` tool was added after a number of students experienced difficulties in constructing two equal lengths emanating from the same point (this requires only two inputs and the original compass tool needed three). The circle tool, located in the creation menu could be used for this purpose, but the students tended to associate the function of this tool exclusively with the production of circles – indicating the importance of the name and location of tool in framing users' operations with it.

a: the symmetrical quadrilateral

b: checking properties with measures

c: checking properties with construction tools

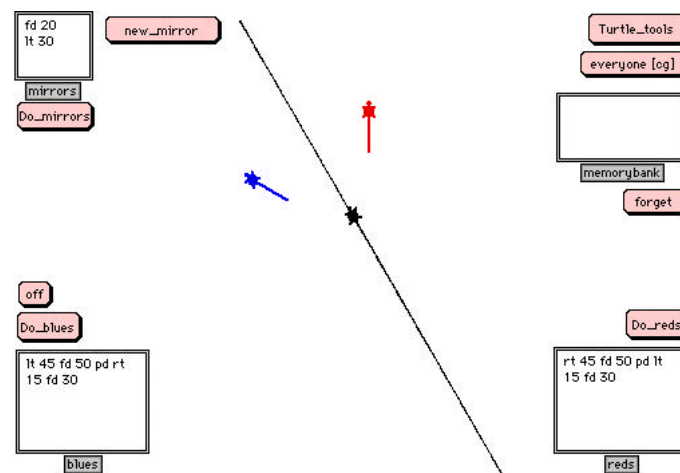**Figure 2: Exploring the symmetrical quadrilateral**

When the task changed from exploration to construction, students' focus also shifted from the quadrilateral to relationships between its vertices – a shift from intra to interfigural analysis – and the activity became that of modelling the behaviour of the `symmetrical point` tool. All pairs were able to enact properties sufficient to define a reflective image point, however most did not manage to formalise these in ways that would produce a general tool for reflection. The idea had been that students would make use of the properties identified in the first part of the task, but in practice this occurred in the activities of only two pairs – the pair who constructed Figure 2c worked out that they could construct their own image point as the intersection point based on any two of construction lines added during their checking activities, coming up with four methods that could be transformed into macro-tools for reflection; and a pair who created two segments connecting point *A* to the axis then added two segments the other side of the axis which were dragged until each one was equal to its corresponding segment.

The rest of the students switched their attention to the perpendicular relationship of point and image with the axis of reflection. The `symmetrical point` tool continued to have an important mediating role – although the students understood they had to build the image point without applying this tool, they tended to leave the now "illegal" point on screen, seeking to find an object which passed through the image-point without actually using it as part of the construction. Once they had this – a line that passed through the point *A* and perpendicular to the axis of reflection – on screen, students knew that the image point should be located along this line at a distance from the axis equal to that of the pre-image point. By this point in their activities, the students appeared to have appropriated the idea of treating points as general dynamic objects with variable positions, rather than specific instances. This meant they were not entirely satisfied with an image point that required some manual

adjustment. They wanted to build a robust construction and they knew which properties to include. What they did not seem to know, despite their previous experiences with the `compass` tools, was which DEG tool could be used to construct the equal distances they wanted – in a way that its equality was maintained under dragging – hence, ending up with a construction that was only semi-robust.
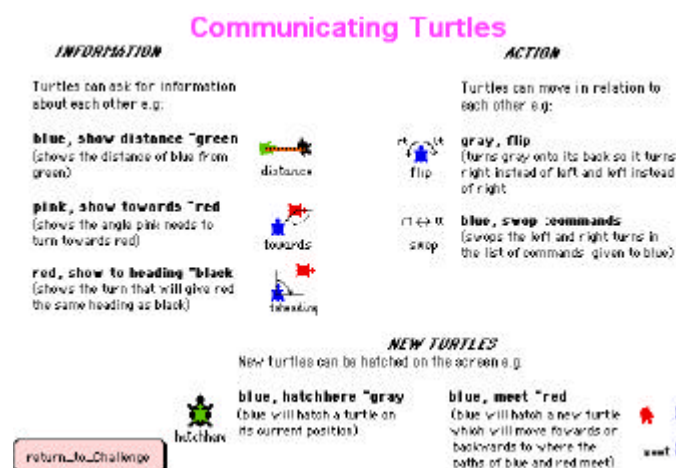
Interactions with the MTG microworld

Figure 3 presents a page from the MTG microworld, which inherited many features from its predecessor Turtle Mirrors (see Hoyles & Healy, 1997).

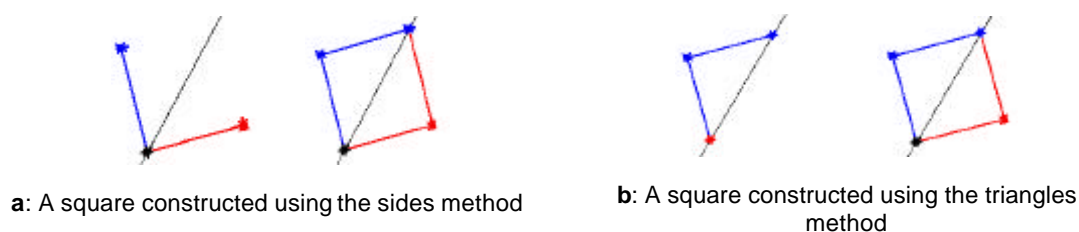**Figure 3: A page from the MTG microworld**

Multiple turtles were used to represent both 'points' – with a blue turtle generally used to draw the pre-image and a red turtle the image – as well as the axis, drawn by the turtle named `mirror`. These turtles could be communicated with via on screen textboxes (as well as the command centre characteristic of all Logo versions). The textbox `memorybank` was also present on screen to provide, along with a tool named `remember`, an accessible way for students to record and recall the information returned by the communication tools. `Remember` had the effect of assigning the output of a command to a variable and displaying this variable on screen in the `memorybank`. Buttons on the screen could be pressed to run a command or a procedure and so facilitate the easy management of the workspace.

In addition to the various screen artefacts, a number of Logo commands were constructed, modified or simply highlighted in a tool-kit presented to students (although students were also free to use any other Logo commands they chose). These are shown in Figure 4.

Figure 4: The MTG tool-kit

In contrast to the DEG microworld, producing the symmetrical MTG quadrilateral proved to be a considerable challenge. Nonetheless, in the two MTG groups who participated in the comparison phase, all pairs eventually managed to construct a sequence of commands that could be used as a template for any symmetrical quadrilateral – without exception, this sequence was initially constructed for the case of the square. Two strategies emerged in both systems, with the *sides method* (building the sides of two symmetrical triangles simultaneously) as shown in the example presented in Figure 5a more common than the *triangles method* (building a triangle and then its image as shown in Figure 5b). Both these strategies made use of two important general – and clearly turtle-mediated – ideas related to reflection: symmetrical figures result when two turtles, with the same starting point, follow paths that are identical except that the direction of turns is swopped; and when a turtle and its image move forward (or backward) they meet on the mirror-line. These generalisations emerged during the design phase and motivated the design of specific tools by which they could be operationalised (swop and meet). The properties emphasised in both these abstractions are the equal pairs of adjacent line-segments (fd or bk command) and angles (turns).

a: A square constructed using the sides method   **b**: A square constructed using the triangles method
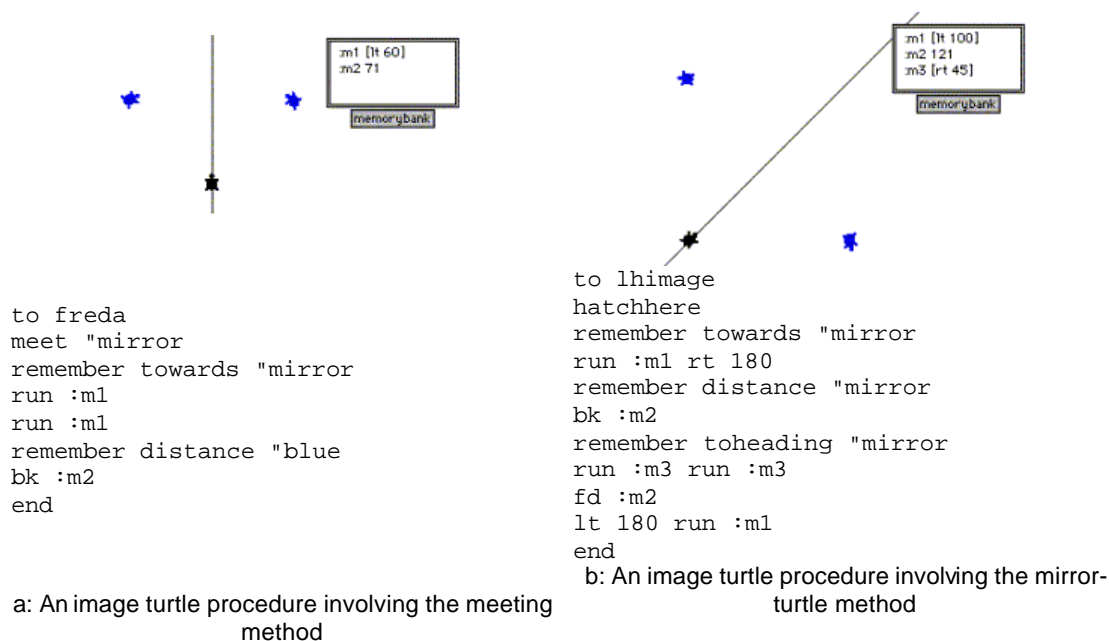
Figure 5: Two MTG methods for constructing a symmetrical quadrilateral

The same properties, guided by the same mathematical generalisations, were evident as students attempted to construct a reflection tool. During students' intrafigural analyses, the asymmetrical relationship between independent variable

(pre-image turtle) and dependent variable (image turtle) that characterises a functional approach to reflection is not made evident – indeed part of the attraction of the turn-swopping abstraction as far as students were concerned was the symmetry in the symbolic code. But the task constraint forbidding communication with the red turtle allowed students to experience in action how the position of an image turtle was dependent on the pre-image turtle. This also helped students to shift their attention towards interfigural analyses and provoked connections of turtle behaviour to emotional as well as spatial connotations – the red turtle was described variously as deaf, dead or asleep. These connections helped the majority of students to shift from thinking about the turtle paths that produced the quadrilateral to relationships between turtles. As they made this shift, the question became, what general method sends a turtle from the pre-image turtle-state to the image turtle-state?

In order to produce what can be considered a reflection function, again two main strategies were observed, distinguishable by the way a new (image) turtle was sent to the axis. The *meeting method* involved starting by sending this turtle to the intersection point of the pre-image turtle and the mirror turtle (an example of one pair's construction is shown in Figure 6a), while in the *mirror-turtle method* the new turtle was positioned on top of the mirror turtle during the construction procedure (see the example in Figure 6b).


```
to freda
meet "mirror
remember towards "mirror
run :m1
run :m1
remember distance "blue
bk :m2
end
```

a: An image turtle procedure involving the meeting method

```
to lhimage
hatchhere
remember towards "mirror
run :m1 rt 180
remember distance "mirror
bk :m2
remember toheading "mirror
run :m3 run :m3
fd :m2
lt 180 run :m1
end
```

b: An image turtle procedure involving the mirror-turtle method

Figure 6.30: Two methods for producing an image turtle

By this stage in the learning system, a MTG-mediated meaning for generality had evolved, with students knowing that a valid construction method was one whose structure could be re-used with specific changes made. Only half the student pairs actually turned the command sequences they used into procedures – and only one of

these worked for all turtle states. The rest of the students worked only in direct drive, although all groups tested that the commands they used worked for different positions of the blue turtle, with those who did not write procedures changing the values in the command sequence as necessary.

MICROWORLD-MEDIATED MEANINGS

For both the DEG and MTG microworlds, students' interactions during the task suggested that they were incorporating the use of interfigural as well as intrafigural analyses and directing their attention to functional aspects of reflection. However, the microworld features that afforded movement from working with reflection as property in the first part of the task to reflection as function in the second as well as the specific aspects of the function notion that were emphasised varied according to which microworld was used. Among the students using DEG tools, the most common representation was as a correspondence relationship based on perpendicular distances, whereas MTG tools were associated with expression of reflection as a mapping of one set of turtles onto another and emphasised equal turns and distances. In terms of the distinctions of the function concept suggested by Sierpinska (1992), the MTG tools appear to emphasise functions in a dynamic form, as processes that map an element of the domain (the independent variable in the case being a member of the turtle-set) onto an element of the range (the dependent variable). In contrast, activities with the DEG microworld seem to prioritise the static form of functions, as relationships or functions between the two sets, domain and range.

Particular features of each microworld were fundamental in supporting students to see and investigate the generality behind the geometrical figures they were producing. Both microworlds afforded views of particular cases as generic examples – in the DEG systems, its was the dragging facilities along with the `symmetrical point` tool that played an important role in signalling geometrical dependencies, whereas the MTG students came to see both turtles and their paths as representative elements of general sets and the (re-)use of symbolic command sequences was fundamental in this respect. However, results also indicate that the same support systems that illuminated general relationships also allowed students to find ways of expressing this generality not through formalisation as intended, but through action. As the students were relative novices with both microworlds, it is possible that this tendency is one that diminishes with time. This is a question for further research.

As a final observation, the analysis of students' interactions suggests that the construction of mathematical meanings for reflection, at least with the DEG and MTG microworlds, involves the forging of connections between general models of reflection and physical movements on screen. In addition, there are some indications that students became particularly engaged with activities into which they could 'import' experiences related to their social practices outside of the mathematics classroom – tools like `meet` and `swop`, for example, were more easily appropriated into student activities than the `compass` tools. A possible interpretation of this finding is that tools are most likely to evoke the intended mathematical knowledge in tasks in which the computational objects built on screen represent a model of an imaginary social situation as well as a physical and a mathematical one.

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