Computer Experiments with Newton's Method

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Newton's method has served as one of the most fruitful paradigms in the development of complex iteration theory.
H.-O. Peitgen, 1988

Abstract

We study the chaotic behaviour of Newton's method with graphic calculators and computers. Through repeated experiments the students can explore the behaviour of the sequence $x_{n+1} = f(x_n) = x_n - f(x_n)/f'(x_n)$ where $f(x)$ is a quintic polynomial. If we expand our study of the dynamics of Newton's method to the complex plane, we find lots of interesting properties: fractals, chaos, attracting periodic cycles and Julia sets. Our study illustrates a symbiotic relationship between technology and mathematics. Technology is used to develop our intuition, and mathematics is used to prove our intuition is correct. Much of what is known about dynamical system was discovered using technology, and it is natural to use technology to study the Newton's method. Almost of our examples were done with the graphic calculators Voyage 200.

Introduction

Since graphic calculators and computers are readily available to most students, now is an especially auspicious time to introduce students to Newton's method as discrete dynamical systems. The process of iteration is impossible to carry out by hand but extremely easy to carry out with a calculator or computer. A very simple six or seven line program allows a student to compute hundreds and thousands of iterations of a single function. Students get feeling that they have the power to explore the uncharted wilderness of the dynamics of Newton's method, not to mention the many, many other simple functions whose dynamics are less well understood. This is a radical new development in mathematics instruction. It gives mathematics an experimental component, a laboratory. Much as the physicists, chemists and biologists have long used the laboratory as an essential component of their introductory courses, now we in mathematics have the same opportunity, and the results should be a much higher appreciation for and recognition of the importance of research mathematics by contemporary students.

The search for solutions of the equation $f(x) = 0$ is ancient. However, it was shown in the early of nineteenth century that there is no general method for solving polynomials of degree five or higher. Consequently, methods for estimating solutions of equations as simple as polynomials are necessary. Newton's method is generally introduced as a useful tool for finding the roots of functions when analytical methods fail. So we will use this method to solve the quintic equation by numerical methods. Some teachers think that just because

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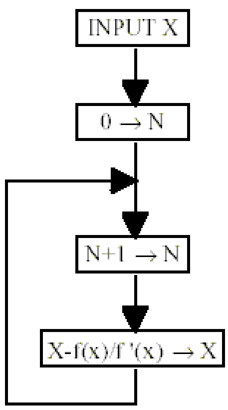
of computers this topic become irrelevant as a subject of learning. And sometimes they ask us: why students should learn such (sophisticated) topics/algorithms in the age of computers? We present some reasons for using Newton's method:

1. The first is that Newton's method is the most common method to solve equation. This method is generally introduced in Calculus I courses as a useful tool for finding the roots of functions when algebraic methods fail. Polynomials used for Calculus I problems do not tend to have any further complications, but polynomials with interesting behavior exist. When we look inside the method, that is, when we consider this iterative algorithm as a discrete dynamical system, a lot of interesting questions appear as we will see.

2. The second reason is that "the behaviour of the Newton map for a polynomial is far from well being understood" [Bal00, p.10]. The dynamics of Newton's method always presents difficult problems, even as applied to polynomials in one variable. The chaotic behaviour of Newton's method is a source of investigation as much for pure mathematics as for the classroom with the help of a calculator/computer. When students learn that this system is not completely understood or that the mathematical ideas they are using were developed in their lifetimes, they change their opinions on the nature of mathematics. Rather than a collection of tricks from centuries past, mathematics becomes an alive and thriving discipline. We should always strive to give our students a glimpse of what is new and exciting in mathematics, and we take that opportunity in our approach.

The basic algorithmic knowledge as a part the teaching and learning of mathematics is necessary for drawing the fractal associated to the Newton's method. We think a student learn better a difficult concept when he/she studies, with some detail, significant examples or problems. We will present several interesting fractals that cannot be fully understood without a use of technology.



Through repeated experiments the students can explore the behaviour of the sequence x_n where $x_{n+1} = N_f(x_n)$ is the Newton map for the quintic function *f*(*x*).

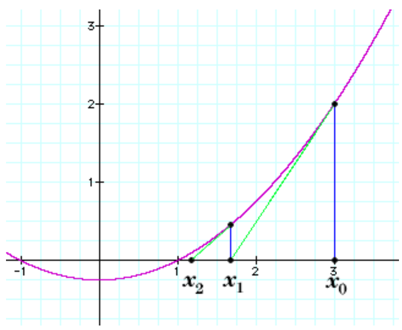
With the current technology there are many approaches that can be taken toward the representation and understanding of a numerical solution to a problem. Generally, the numerical representation of a solution to a problem is a table of solution values, which may be exact or approximate. We present three approaches that current technology enhances: (*i*) the graphical or geometric approach, (*ii*) the discrete dynamical system (or sequence) approach, (*iii*) the programming approach. The graphical approach aids in student understanding of approximation. The TRACE and TABLE modes of the graphical calculator interface the general picture of the graph and provide the user with the specific numerical values. Most algorithms in numerical analysis can be written as a discrete dynamical system or as a system of difference equations. We have found that this discrete approach to modelling is intuitive to students and with minimal introductory work is accessible to most students. Finally, a more traditional approach to numerical solutions is to write programs for the algorithms as they are developed. This is a less accessible aspect of the calculator, but very valuable to students for understanding computer logic and the student ownership of developing their own programs. We presented all these approaches at our work and encourage teachers to choose the approach that best suits their course objectives.

In the Madeira Island (Portugal) a study on the integration of graphic calculator in mathematics education was carried out by the first author [Prog97] since 1997. We introduce Discrete Dynamical Systems in this project in 1998.

We include some exercises and strategic activities for students.

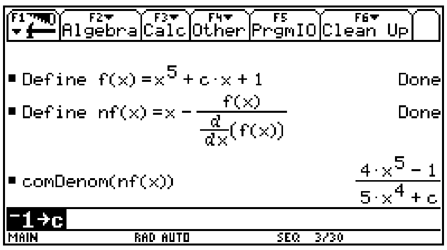
Newton’s method as a discrete dynamical system

We begin with the basic ideas and terminology of iteration. Given an initial guess x_0, we approximate the function by a tangent line passing through the point $(x_0, f(x_0))$ with gradient $f'(x_0)$. This has the form $y = f'(x_0) \times x_0 + c$ which is satisfied by the point $(x_0, f(x_0))$. This implies $f(x_0) = f'(x_0) \times x_0 + c$ and $c = f(x_0) - f'(x_0) \times x_0$. The equation of the tangent line is therefore $y = f'(x_0) \times x + (f(x_0) - f'(x_0) \times x_0)$.



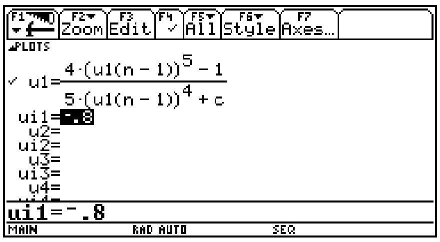
Solving for the root of the tangent line, which we denote x_1, we have $0 = f'(x_0) \times x_1 + (f(x_0) - f'(x_0) \times x_0)$ and $x_1 = x_0 - f(x_0)/f'(x_0)$, as we se in the figure. We can find successive approximations $x_1, x_2, ..., x_n, ...$ to the root by iterating this process and applying the relation $x_{n+1} = N_f(x_n) = x_n - f(x_n)/f'(x_n)$. Notice that if x_n is an exact root of $f(x)$, we have $f(x) = 0$, and consequently, $x_{n+1} = N_f\left(x_n\right) = x_n$.

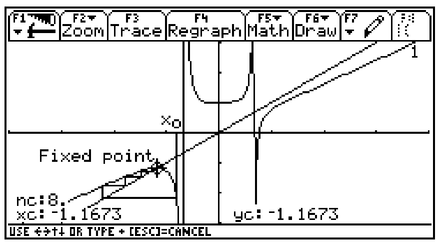
Students need to be provided with opportunities for practice and reflection on solving significant problems. In this respect, technology may play an important educational role, changing the focus from mechanical and repetitive process to the comprehension of algebra and calculus as instruments that enable the modeling of real situations.



Newton map for the quintic $f(x)=x^5+cx+1$

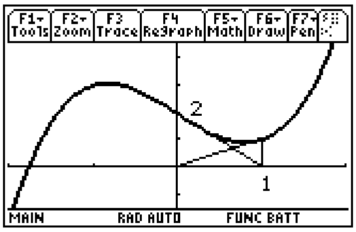
For intermediate calculations we can use a calculator with CAS as Voyage 200. The actual calculators possess the *web* type for graphics.





The process of using the calculator to find the iterations helps students get a good numeric sense of how Newton's method moves from a starting value to a root of an equation. We use $N^n(x) = N \circ N \circ ... \circ N(x)$.

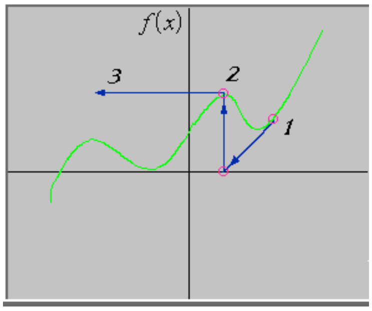
It is possible to have stable periodic points, i.e., $N^n(x_0) = x_0$ for some $n \geq 2$ $(n \in IN)$ and $|(N^n)'(x_0)| < 1$? To answer this question we present one concrete example with periodicity 2 in the iteration of Newton's method for $f(x) = x^3 - 2x + 2$. N_f has the 2-cycle 0→1→0 for *f*(*x*)= x^3 -2x+2 that gives $N_f(x)=x-(x^3-2x+2)/(3x^2-2)$.



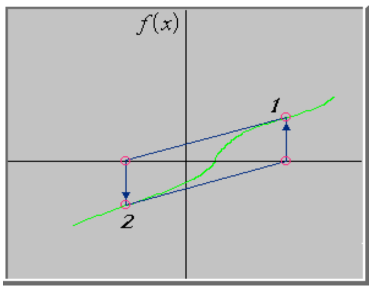
It turns out that it is possible, so how can we find them?

There are two kind of possible difficulties in the Newton map as we see in the next figures:

1. The simplest way that the sequence $N_f^n(x)$ may fail to converge to a root of *f* is when this sequence becomes undefined at some finite stages, i.e., when some iterate of N_f maps *x* onto a vertical asymptote of N_f, i.e., when $f'(x) = 0$. By the mean theorem this type of nonconvergence must occur whenever *f* has more than one real root.



2. The next simplest way that nonconvergence of $N_f^n(x)$ can occur is when *x* is periodic or eventually periodic under N_f: *x* is *periodic* if $N_f^k(x) = x$ for some *k* bigger than one, its *period* is the least value of *k* which this hold; *x* is *eventually periodic* if $N_f^{m+k}(x) = N_f^m(x)$, with $k \geq 2$.



We conclude that Newton's method can fail and we can observe that we must be careful with iteration in other numerical methods. Indeed the dynamics of Newton's method always presents difficult problems, even as applied to polynomials in one variable. For instance, already for quintic polynomials there may be open sets of initial points which do not lead to any root but instead to an attracting cycle of length greater than one, and the boundaries of the basins will usually be complicated fractals whose topology is poorly understood. Here we study this aspect graphically.

A key concept in the study of discrete dynamical systems is that of *chaos* or *sensitive dependence on initial conditions*. There have been several definitions of chaos, for example, in [Dev89]. With computer experiments we study the existence of chaotic behaviours in the iteration of Newton's method for quintic functions. Probably the most important advantage is that this model is very intuitive and easy understood by students at many levels. Iterating this recursion relationship can provide quick insight and many times answers to problems with no analytical solution.

In first place we remember that the general quintic equation $x^5 + c_1 x^4 + c_2 x^3 + c_3 x^2 + c_4 x + c_5 = 0$, with arbitrary coefficients c_i, can be transformed to the Bring-Jerrard type $x^5 + a_1 x + a_2 = 0$ by a Tschirnhaus transformation. The a_i can ultimately be expressed in radicals in terms of the c_i [Wei99], [FSR03a]. However the resulting expressions are typically enormous. For a general quintic with symbolic coefficients they require a lot of storage. But this is not a problem in the computer era. And with some algebraic calculus and topological equivalence we can conclude [FSR03b] that the most interesting case for Newton map of the quintic $x^5 + a_1 x + a_2 = 0$ is for $f(x) = x^5 + cx + 1$. So we concentrate our computers experiment in the Newton's method for the quintic polynomial of the form $f(x) = x^5 + cx + 1$.

Much of the motivation for the material to be presented here is the following theorem due to P. Fatou [Pei86, p. 27]:

Theorem: Let $N(x)$ *be a rational function with a stable periodic cycle (*$\exists x_0 : N^n(x) = x_0$*, i.e., the orbit* $\{x_0, N(x_0), N^2(x_0), ..., N^{n-1}(x_0)\}$*, with* $n \geq 2$*), then such orbit can be obtained as the limit of successive images* $N_f^n(x_c)$*, as* $k \to \infty$*, where* x_c *is a critical point of* $N(x)$*, i.e.,* $N'(x_c) = 0$*.*

In our case we have $N'_f(x) = 20x^3 f(x)/(f'(x))^2$, so the critical points of N_f are the zeros of $f(x)$ or x=0. We remember that the zeros of $f(x)$ are also fixed points of N_f, so for the iteration of N_f we start on the free critical point x=0.

Let us now describe the numerical experiments which are performed in the c-parameter plane. The *bifurcation diagram* is a record of the eventual orbit values (plotted vertically) for each value of the parameter c belongs to the an interval (plotted horizontally). This gives a record of how the dynamics change as the parameter varies.

We need programming but with only few lines as we see in the right box. In this perspective the new technology is good. We use the Voyage 200 and the software *Mathematica*®. The student also has opportunities to write meaningful computer programs to test a growing theory. We find it appropriate to begin computer work by giving to the class a program with which everyone can begin to experiment.

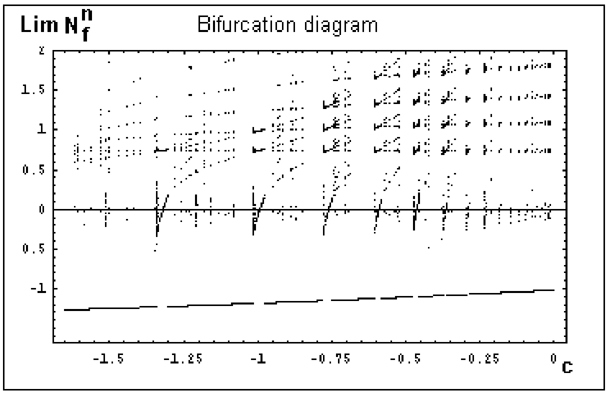
input Cmin, Cmax
$f(x) = x^5 + c\,x + 1$
0→x
$N_f(x_n) = x_n - f(x_n)/f'(x_n)$
for c=cmin **to** cmax
  for i=1 **to** 150
    $N_f(x) = x$
  if i>120 **then** **print** (c, x)

```
Clear[c, f, Nf, x]; lista = {}; cmin = -1.65; cmax = 0;
f[x_] := x^5 + c x + 1;  Nf[c_, x_] = x - f[x] / f'[x];
For [c = cmin, c < -cmax, c += 0.0001,  x = 0.0;
        Do [ x = Nf[c, x];
        If [j > 100,  AppendTo[lista, {c, x}]], {j, 1, 250}]
      ];
 ListPlot[ lista, AxesOrigin -> {Co, 0}, PlotRange -> {-2, 2},
 PlotStyle -> PointSize [0.005], Frame -> True];
```

Program for Bifurcation diagram in *Mathematica*

The program for Voyage200 is very similar to the pseudocode presented above.

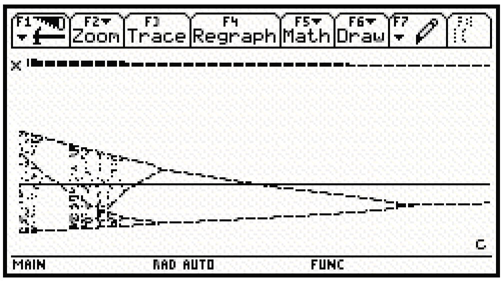
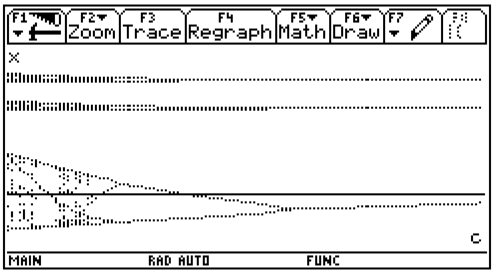
In the bifurcation diagram the point $\left(c, N_{f_c}^{n}(0)\right)$ are plotted. In our case $N_{f_c}^{n}(x)$ is the Newton map for $f(x) = x^5 + cx + 1$. We use parameter values *c* between -1.649 and 0.



We can see chaotic behaviour in the left graph where there is all kind of periodic point. With change $c \in [C\min, C\max]$ in the previous program, we can see better the periodic point.

For this kind of graph we can use the graphic calculator as we exemplify below.

We experiment various chance in parameter *c* near -1.33 (period 2) and near -1 (period 3) and we obtain similar graphs, this is what we call self-similarity. This time we use the calculator Voyage 200.

Periods **2**, 2^2, …, 2^n …	Periods **3**, $3\times 2, 3\times 2^2$, …, 3×2^n …
	
$[-1.34,-1.33]\times[-0.4,0.8]$	$[-1.1,-1.0]\times[-0.3,1.1]$

We found period three and period three implies chaos [LY75].

Indeed for these bifurcation diagrams it is very laborious using the graphic calculator because the looping in the program for iteration and so it is not appropriate to use in the classroom. We suggest preferentially the use of computer in the classroom instead of graphic calculator however students at home can use graphic calculator. The Voyage 200 is similar with the TI-89/92.

Students can easily investigate in the classroom the chaotic behaviour of Newton's method with computer graphics as a tool. Since computers and graphic calculators can readily graph approximate solutions, students must be prepared to interpret what they see and evaluate the validity of their computations.

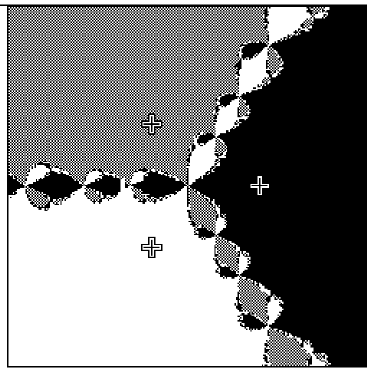
The computer is an instrument of data processing, and the concept of "data" seems to denote numbers, not pictures. In actual fact, however, pictures are just another means of describing content: the results are called computer (generated) graphics and the mathematicians instead of giving an abstract presentation in so many dry word, they have chosen pictures as we exemplify here. It is important that we teach our students to think graphically as well as analytically since graphical illustration of quantitative information is much more common in today's society.

Julia Sets

If we expand our study of the dynamics of Newton map to the complex plane, we find lots of interesting properties. Fractals, chaos, attracting periodic cycles, Julia sets, and other phenomena are present, depending on what functions we study. Although Newton's method is an old application of calculus, it was discovered relatively recently that extending it to the complex plane leads to a very interesting fractal pattern. The study of Newton's method in the complex setting was initiated by E. Schröder (1870/71) and A. Cayley (1879), for quadratic polynomials [Pei91, p. 356]. They where able to study the quadratic case $z^2 - 1 = 0$. Cayley commented that "The solution is easy and elegant in the case of quadratic equation, but the next succeeding case of the cubic equation appears to presents considerable difficulty".

We can understand what Cayley conjectured with the Julia's work and Fatou's work in the beginning of the XX century. The immense progress that Julia and Fatou were able to make must be valuable all the more because in those days there were no computers to aid in the understanding of the complicated mater; instead they had to rely completely on their imagination.

But nowadays in the classroom we can understand what Cayley conjectured with the help of a graphic calculator or computer. The figure at right exemplifies the complexity of Newton map for the equation $z^3 - 1 = 0$ studied by Cayley.



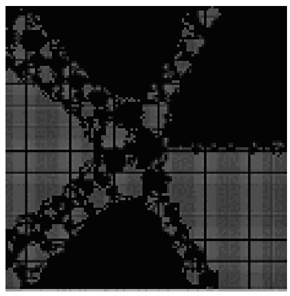
Julia set for $z_{n+1} = z_n - \frac{z_n^3 - 1}{3z_n^2}$ in the complex plane. Each region converges for one of the 3 solutions of $z^3 - 1 = 0$ represented by the crosses.

These last figures are good examples of a theme that is easily computed by students in an introductory numerical course and that leads to interesting, and largely unexplored, questions that might entice a student to get interested in numerical analysis.

The Julia set is the set of repelling periodic points of the map and their limit points.

The Julia set is invariant under the action of the map and has chaotic dynamics. Trying to infer the structure of these boundaries is very difficult without a computer.

With this example students discover how Newton's method can lead to a simple fractal. We can do the same for the Newton map of the quintic function. The picture at right represents the Julia set for Newton map associated to $f(z) = z^5 - z + 1$.



Activities for students

1. Exercises with calculator/computer

With the help of a calculator/computer try to solve the next exercises.

1. Let $f_c(x) = x^5 + cx + 1$. Find a value of *c* for which there is an interval I in which the Newton map $N_f(x)$ has periodic points of all periods.

Observation: Investigate the *Sharkovsky order*.

2. Let x_0 and *A* be positive real numbers and consider the sequence defined by

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{A}{x_n}\right) \quad \text{(\textit{Babylonian algorithm}).}$$

a) Find a function *f* such that $x_n = N_f{}^n(x_0)$ (*Newton's method*).

b) What is the limit of the sequence x_0, x_1, x_2,..., in case that exists?

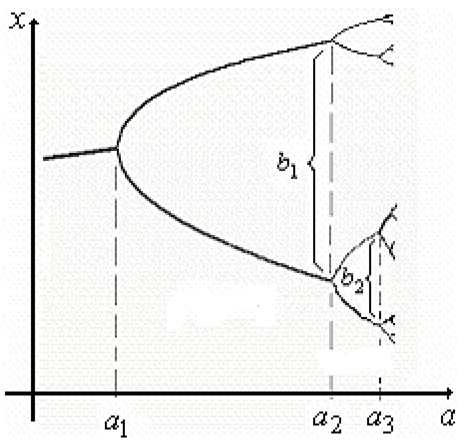
c) Apply this algorithm for the calculation of $\sqrt[n]{A}$ with $n > 1$.

Note: This algorithm is often used in square root routines for computers.

2. Student Project: Feigenbaum (Universal) Constants

Investigations of the distances between successive period-doubling bifurcations led to the discovery of a new kind of scaling, and a new mathematical constant that is universal in the sense that it arises in a large class of functions.

Let *f*(*x*) a quadratic polynomial; we exemplify with $f_a(x) = ax(1-x)$. In the right figure the a_n is such that $(f^n)'(a_n) = -1$, i.e., a_n is a period doubling bifurcation point. And for the sequence b_n see the figure at right.



For the quadratic function, we have

$$\delta = \lim_{n\to\infty} \frac{a_{n+1} - a_n}{a_n - a_{n-1}} = 4.6692216091... \quad \text{and}$$

$$\alpha = \lim_{n\to\infty} \frac{b_{n+1}}{b_n} = 2.5029078750...$$

With the help of your calculator/computer try to calculate approximately the two limits $\delta = \lim\limits_{n\to\infty} \frac{c_{n+1} - c_n}{c_n - c_{n-1}}$ and $\alpha = \lim\limits_{n\to\infty} \frac{b_{n+1}}{b_n}$ for the Newton map of the quintic $f(x) = x^5 + cx + 1$, i.e., for $x_{n+1} = N_f(x_n) = x_n - f(x_n)/f'(x_n)$.

Observation: You must investigate more about Feigenbaum Constants.

Conclusion

Here we present new material. But the first author used part of these activities in the cubic equation during a semester for secondary teachers in 1998/99 and in 1999/00 he collaborates with these teachers and they used some of his activities with their students in their secondary school [BCGL00]. In all projects of the first author [Pro97] he emphasise the change from traditional pencil and paper skills to a technology-supported modelling approach and he uses calculator with CAS. The second author always presents the Newton's method in his course at university and he uses the Mathematica® software.

It is worth including dynamical systems in our courses? Our example illustrates a symbiotic relationship between technology and mathematics. Technology is used to develop our intuition, and mathematics is used to prove our intuition is correct. Much of what is known about dynamical system was discovered using technology, and it is natural to use technology to study the dynamical systems and in particular the Newton's method. It is also important to recognize the fact that many of the standard computational techniques that are part of ours courses are embedded in computer-base mathematics systems as *Derive* (TI-89/92) and Mathematica®. This gives us the freedom to concentrate more on the underlying mathematics and less on the symbolic manipulation.

One of the fundamental uses of computer graphics is in science and education. Iterating a function is a simple process, yet the results are often very complex. Students also see Newton's method in a new light and are surprised and fascinated by the intricacies of the dynamics. The computer offers us the possibility of experimentation: student can check the influence of parameters in the Newton's method, the result of transformations, and the limiting values of iteratively applied calculation.

One might also wonder why we could choose to use a method that is subject to problems like those just demonstrated. The reason is speed: Newton's method exhibits quadratic convergence first proved by Fourier in 1818. Essentially, this means that the number of correct digits to the right of decimal place in our estimate doubles with each iteration of Newton's method.

We found the investigations of fractal and chaos to be a marvellous topic for secondary school and for the university. In addition it is suited for the discovery approach to learning and the incorporation of technology. These activities allow the mathematics teacher and students to see new topics in mathematics curriculum which are more complicated without technology, namely discrete dynamical system. Dynamical systems is currently one of the most actively researched branches of mathematics. Applications to modelling the weather, the central nervous systems, and the stock market suggest the intrinsic nature of dynamical systems.

In the mathematics education reform at Chalmers [Lar01, p. 6] they also include the Newton's method and they suggest that "each student implements Newton's method in Matlab".

One of the aims in teaching mathematics should indeed be to "give the young a feeling for the beauty and eloquence of mathematics and its profound relationship with the real world", and it is difficult to see how the mathematics teacher can ignore an aspect of mathematics that is accessible with quite elementary mathematical ideas: fractal and chaos.

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